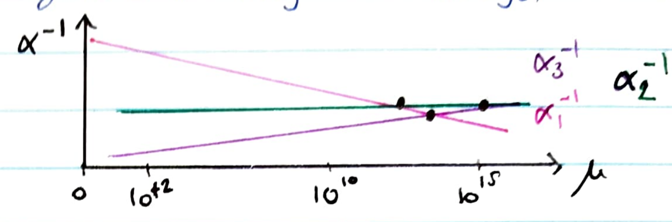
CH5 GRAND UNIFICATION

→ The current gauge theory of the standard model is

$$SU(3)_c \times SU(2)_L \times U(1)_Y$$

with coupling $g_3(\mu)$, $g_2(\mu)$, $g_1(\mu)$ with μ the energy scale.

→ Since the couplings are running with energy, one can show that:



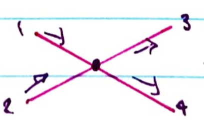
→ Usually, $\alpha_1 = g_1^2/4\pi$. Now, in the GUTs theories, the normalization changes du to the embedding of $U(1)_Y$ into a larger unified gauge group (ex: $SU(5)$ or $SO(10)$), so that $\alpha_1 = \frac{5}{3}\frac{g_1^2}{4\pi}$

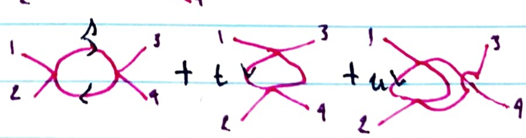
→ To derive the β-function, one can either use EFT's or quantum loops (the β-function arises from the renormalization scale dependance of the coupling).

5.1 Running of $\lambda\phi^4$

→ As an example, let us focus on the following action:

$$\mathcal{L} = \frac{-1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$$

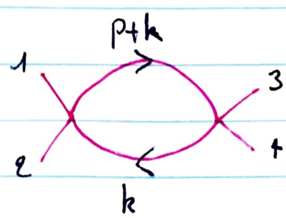
→ At first order, $i\mathcal{M} =$  $= -i\lambda$

→ At 1 loop order, $i\mathcal{M} =$ 

DEF | Convenient variables are the Mandelstam variables s, t, u defined as

$s = (p_1+p_2)^2$ $\quad t = (p_1-p_3)^2$ $\quad u = (p_1-p_4)^2$

→ We compute the s-channel:



$$i\mathcal{M} = \frac{(-i\lambda)^2}{2}\int \frac{d^4k}{(2\pi)^4} \frac{i}{(p+k)^2-m^2} \frac{i}{k^2-m^2}$$

Using Feynman parametrization: $\frac{1}{AB} = \int_0^1 dx\,(xA+(1-x)B)^{-2}$, we can write:

$$i\mathcal{M} = \frac{\lambda^2}{2}\int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \left(x[(p+k)^2-m^2] + (1-x)(k^2-m^2)\right)^{-2}$$

Now, $x[(p+k)^2-m^2] + (1-x)(k^2-m^2)$

$= xp^2 + xk^2 + 2xpk - xm^2 + k^2 - m^2 - xk^2 + xm^2$

$= xp^2 + 2xpk + k^2 - m^2$ and calling $\ell \equiv k + xp$, one has

$= \ell^2 - x^2p^2 + xp^2 - m^2$ $\qquad \ell^2 = k^2 + x^2p^2 + 2kxp$

$= \ell^2 + (x-1)xp^2 - m^2 = \ell^2 - (m^2 - xp^2(x-1))$. Let $\Delta \equiv m^2 - xp^2(x-1)$

$$i\mathcal{M} = \frac{\lambda^2}{2}\int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4} (\ell^2-\Delta)^{-2}$$

→ Now, we use Wick rotation to go to Euclidean space:

peskin A.44

$$i\mathcal{M} = \int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{(\ell^2-\Delta)^2} = \frac{\lambda^2}{2}\int_0^1 dx \frac{i}{(4\pi)^{d/2}} \frac{\Gamma(2-d/2)}{\underbrace{\Gamma(2)}_{=1}}\left(\frac{1}{\Delta}\right)^{2-d/2}$$

$$= \frac{i\lambda^2}{2(4\pi)^{d/2}}\Gamma(2-d/2)\int_0^1 dx \left(\frac{1}{\Delta}\right)^{2-d/2}$$

$$\lim_{d\to 4}(i\mathcal{M}) = \frac{i\lambda^2}{2(4\pi)^{2-\epsilon}}\Gamma(\epsilon)\int_0^1 dx\,(1/\Delta)^{\epsilon}$$ where we wrote $d = 4-2\epsilon$

Now, at 1st order, $\Gamma(\epsilon) \simeq \epsilon^{-1} - \gamma$ and $(1/\Delta)^{\epsilon} = 1 + \epsilon \ln(1/\Delta)$

also, $(4\pi)^{+\epsilon} = 1 + \epsilon \ln 4\pi$

$$i\mathcal{M} = \frac{1}{2}\frac{i\lambda^2}{(4\pi)^2}\int_0^1 dx \left\{\frac{1}{\epsilon} - \gamma + \ln 4\pi - \ln\Delta + \mathcal{O}(\epsilon)\right\}$$

Since $\Delta \sim p^2$, there is a log dependence in the energy.

→ Alternatively, one can use the Pauli-Villars regularization, by introducing a heavy fictitious particle. For example, replace

$$\frac{1}{k^2+i\epsilon} \mapsto \frac{1}{k^2+i\epsilon} - \frac{1}{k^2-M^2+i\epsilon}$$

$s = p$

The term $\frac{1}{\epsilon} - \gamma + \ln 4\pi$ is then replaced by $\ln(M^2 - s.x(1-x))$ so that:

$$\int_0^1 dx\,\{\ln(M^2 - s.x(1-x)) - \ln(m^2 - s.x(x-1))\} \sim \ln\left(\frac{M^2}{s}\right)$$

↳ M is a regulator of the theory.

→ The complete amplitude reads:

$$i\mathcal{M} = -i\lambda + (-i\lambda)^2 \left(iV(s) + iV(t) + iV(u) \right)$$

where $V(\sigma) \equiv \frac{-1}{32\pi^2} \int_0^1 dx \left(\frac{1}{\epsilon} - \gamma + \ln 4\pi - \ln\{m^2 - x(1-x)\sigma\} \right)$

→ We can interpret the latter as a correction to the coupling:

$$\tilde{\lambda}(\mu) = \lambda - \lambda^2 \frac{1}{32\pi^2} \left(3\ln\Lambda^2 - \ln s - \ln u - \ln t \right) + \mathcal{O}(\lambda^3)$$

Inverting the perturbative expansion, one finds:

$$\lambda = \tilde{\lambda}(\mu) + \frac{1}{32\pi^2} \lambda^2 (3\ln\Lambda^2 + \dots)$$

$$= \tilde{\lambda}(\mu) + \frac{1}{32\pi^2} \left(\tilde{\lambda} + \frac{1}{32\pi^2} \lambda^2 (3\ln\Lambda^2) \right)^2 (3\ln\Lambda^2 + \dots)$$

$$= \tilde{\lambda}(\mu) + \frac{1}{32\pi^2} \tilde{\lambda}^2(\mu) (3\ln\Lambda^2 + \dots) + \mathcal{O}(\tilde{\lambda}^3)$$

Now, consider a different scale $\mu' \neq \mu$, so that $t' \neq t$, $s' \neq t$, $u' \neq u$:

$$\tilde{\lambda}(\mu') = \lambda - \frac{1}{32\pi^2} \lambda^2 (3\ln\Lambda^2 - \ln s' - \ln t' - \ln u')$$

$$= \tilde{\lambda}(\mu) + \frac{1}{32\pi^2} \tilde{\lambda}^2(\mu) (3\ln\Lambda^2 - \ln s - \ln t - \ln u)$$

$$- \frac{1}{32\pi^2} \tilde{\lambda}^2(\mu) (3\ln\Lambda^2 - \ln s' - \ln t' - \ln u') + \dots$$

$$\Rightarrow \tilde{\lambda}(\mu') = \tilde{\lambda}(\mu) + \frac{1}{32\pi^2} \tilde{\lambda}^2(\mu) \underbrace{\left(\ln s'/s + \ln t'/t + \ln u'/u \right)}_{= 3\ln \mu'/\mu}$$

→ When dealing with EFT, one wants to understand the evolution of the theory between 2 scales.

→ Let's compute the β-function:

$$\mu \frac{d\tilde{\lambda}}{d\mu} = \frac{3}{16\pi^2} \tilde{\lambda}^2 \equiv \beta(\tilde{\lambda}) > 0$$

Integrating: $\frac{-1}{\tilde{\lambda}} + \frac{1}{\tilde{\lambda}_0} = \frac{3}{16\pi^2} \ln \mu/\mu_0$

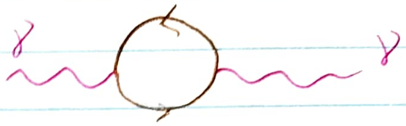
$$\Leftrightarrow \tilde{\lambda}(\mu) = \frac{\tilde{\lambda}_0}{1 - \frac{3}{16\pi^2} \tilde{\lambda}_0 \ln \mu/\mu_0} \simeq \lambda_0 + \frac{3}{16\pi^2} \lambda_0^2 \ln\left(\frac{\mu}{\mu_0}\right) + \mathcal{O}(\lambda_0^3)$$

→ We see that when $\mu = \mu_0 \exp\left\{ \frac{16\pi^2}{3} \cdot \frac{1}{\lambda_0} \right\}$, $\lambda \to \infty$

DEF The energy scale at which the coupling diverges is called the Landau pole. To have a Landau pole, you need

$$\beta > 0$$

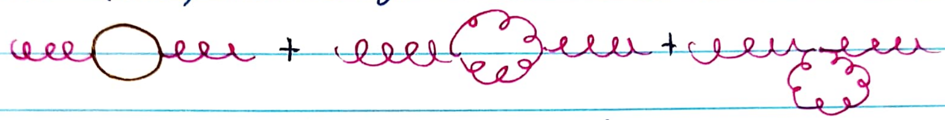
→ In QED, one can show that



$$\alpha(\mu) = \frac{\alpha_0}{1 - \frac{2}{3\pi}\alpha_0 \ln(\mu/\mu_0)} \quad ; \beta > 0$$

Its Landau pole is $\mu_{LP} = \mu\, e^{3\pi/2 \cdot \alpha_0} \sim m_e\, e^{500} \gg M_{PL}$

→ In QCD, more diagrams contribute to the 1-loop correction:



The self interaction act as anti-screening

5.2 Yang-Mills theory (see Peskin, 15.4)

→ A gauge theory with self interaction from the gauge bosons is refered to as a Yang-Mills theory.

→ It is based on a non-abelian group. Consider SU(N) with N_f fermions. The one-loop contribution to the β-function gives:

PROP

$$\mu \frac{dg}{d\mu} = \beta(g) = \frac{-g^3}{(4\pi)^2}\left(\frac{11}{3} C_2[G] - \frac{4}{3} T[R] \cdot N_f\right)$$

where $C_2[G]$ is the quadratic Casimir invariant of the group G

$T[R]$ is the trace normalisation factor of the rep. R of the fermions: $\mathrm{Tr}[T_R^A T_R^B] \equiv T(R)\,\delta^{AB}$

→ In the fundamental rep., $T[R] = 1/2$, and for SU(N), $C_2 = N$.

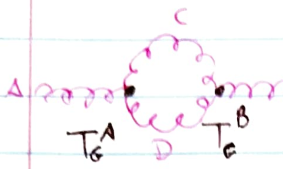
→ In QCD, $N=3$ and $N_f = 6$ (6 quarks), so that

$$\beta(g_3) = \frac{-g_3^3}{(4\pi)^2}\left(\frac{11}{3} \cdot 3 - \frac{4}{3} \cdot \frac{1}{2} \cdot 6\right) = -\frac{7 g_3^3}{(4\pi)^2} < 0 \text{ for } E > m_q$$

→ At each vertex, one can have from A to B gauge field

$\sim \delta^{AB} \sim \mathrm{Tr}[T^A T^B]$

Indeed, the interaction with a fermion has a current
$$J^{\mu,A} = \bar{\psi}_i (T^A)_{ij} \gamma^\mu \psi_j$$



$\sim \sum_{C,D} f^{ACD} f^{BCD} = \sum_{C,D} (T_G^C)_{AB} (T_G^C)_{BD}$

DEF In general, $[T^A, T^B] = i f^{ABC} T^C$. In the adjoint representation G, one has
$$(T_G^A)_{BC} = i f^{ABC}$$

→ Example: $(T_G^A)(T_G^B) = (T_G^A)_{CD} (T_G^B)_{DE}$
and $\mathrm{Tr}(T_G^A T_G^B) = (T_G^A)_{CD} (T_G^B)_{DC} = -f^{ACD} f^{BDC} = +f^{ACD} f^{BCD} = T(G)\delta^{AB}$

→ If we localize the gauge transformation in our lagrangian,
$\mathcal{L}_\psi \mapsto i\bar{\psi} U^{-1} \not{\partial} U \psi - m \bar{\psi} U^{-1} U \psi$ where $\begin{cases} \psi^A \mapsto U^A{}_B \psi^B, \\ \bar{\psi}_A \mapsto \bar{\psi}_B (U^\dagger)^B{}_A \end{cases}$
$\mathcal{L}_\psi$ is not invariant anymore.
We need to introduce a new quantity.

DEF The covariant derivative D is defined by how it transforms:
$$(D_\mu \psi)' \equiv U D_\mu \psi$$
Thus, we introduce a connection A_μ in the adjoint rep.:
$$(D_\mu \psi)^A \equiv \partial_\mu \psi^A - i (A_\mu)^A{}_B \psi^B$$

Notice that $D'_\mu \psi' = (\partial_\mu - i A'_\mu)\psi' = (\partial_\mu - i A'_\mu) U\psi = U \partial_\mu \psi + \partial_\mu U \cdot \psi - i A'_\mu U \psi$
$U D_\mu \psi \overset{!}{=} U \partial_\mu \psi - i A_\mu U \psi \Leftrightarrow A'_\mu = -i \partial_\mu U \cdot U^{-1} + U A_\mu U^{-1}$
Going in the algebra: $= i U \partial_\mu U^\dagger + U A_\mu U^\dagger$
$A'_\mu = \partial_\mu \alpha + (1 + i\alpha) A_\mu (1 - i\alpha) = \partial_\mu \alpha + i[\alpha, A_\mu]$ with $U \simeq 1 + i\alpha_a(x) t_a$
In the $\{t_a\}$ basis, $\delta A_{\mu a} t_a = \partial_\mu \alpha_a . t_a + i \alpha_a A_{\mu b} . i f_{abc} . t_c$

→ The covariant derivative reduces to $D_\mu = \partial_\mu - i A_\mu^A T_R^A$

→ In the rep. of $SU(N)$, we have $U^\dagger U = \mathbb{1}$ and $\det U = \mathbb{1}$.
The generators of $su(N)$ then have $(T^A)^\dagger = T^A$ and $\mathrm{Tr}\, T^A = 0$
↳ There are $N^2 - 1$ elements
$SO(3)$: 8 gauge bosons, $SO(2)$: 3 gauge bosons

→ In the fundamental rep F, one has
$$\mathrm{Tr}(T_F{}^A T_F{}^B) = C(F)\,\delta^{AB} = 1/2\,\delta^{AB} \quad \text{and } \dim F = N$$
In the adjoint rep G, one has
$$(T_G{}^A)_{BC} = i f^{BAC}, \text{ so that } [T_G{}^A, T_G{}^B] = i f^{ABC} T_G^C \;;\; \dim G = N^2-1$$

→ Under a representation of SU(N) of dimension r, a field ψ^i transforms as $\psi'^i = U^i{}_j \psi^j$, $i,j = 1,\dots,r$
$$(\psi'^i)^* = (U^i{}_j)^* (\psi^j)^* = (U^+)^j{}_i\, \psi_j \qquad (\psi^j)^* \sim \psi_j$$
Also, $U^+U = 1 \Leftrightarrow U^{+k}{}_i U^i{}_j = \delta^k_j \Leftarrow U^{+k}{}_j = U^{*}{}_k{}^j$

↳ One can build a scalar (an object invariant under the group) as:
$$\psi^+\chi \mapsto \psi'^+\chi' = \psi'_i \chi'^i = \psi_i U^{+i}{}_j U^j{}_k \chi^k = \psi_i \chi^i$$

→ For the fundamental rep. of SU(N), we denote
$\psi^i \sim N$ and $\chi_i \sim \bar{N}$ its conjugate

→ From the fund. rep., one can build objects with multiple indices
ex: $\psi^i \chi^j \equiv \phi^{ij}$ that transforms as $\phi' = U^i{}_k U^j{}_\ell \phi^{k\ell}$
↳ We can build higher rep. using tensor product of the fund. rep.

→ Consider
$$\phi^{ij} = \phi_S^{ij} + \phi_A^{ij} = \phi^{(ij)} + \phi^{[ij]}$$
$$= \frac{1}{2}(\phi^{ij} + \phi^{ji}) + \frac{1}{2}(\phi^{ij} - \phi^{ji})$$

↳ The symmetric part ϕ_S has $N(N+1)/2$ dof and the antisymmetric part ϕ_A has $N(N-1)/2$
→ The symmetry is preserved under transformation:
$$\phi_S'^{ij} = U^i{}_k U^j{}_\ell \phi_S^{k\ell} = U^i{}_k U^j{}_\ell \phi_S^{\ell k} = \phi_S'^{ji} \;,\; \phi_A'^{ij} = -\phi_A'^{ji}$$

→ In SU(3), if $\psi^i, \chi^i \in \underline{3}$, then $\phi^{ij} \in 3 \otimes 3$
Notice $N(N-1)/2 = 3$ and $N(N+1)/2 = 6$. The irrep decomposition of $3 \otimes 3 = \bar{3}_A \oplus 6_S$ where $\bar{3}$ is the conjugate rep and 6 is the symmetric part.
→ In SU(2), $2 \otimes 2 = 1_A \oplus 3_S$

⊙ SU(N) invariants:

1) Notice $U^\dagger \mathbb{1} U = \mathbb{1}$ and $(U^\dagger)^i{}_j \, \delta^j{}_k \, U^k{}_\ell = \delta^i{}_\ell$

→ $\delta^i{}_j$ is an invariant tensor under SU(N)

2) $\det U = \mathbb{1} \Rightarrow \varepsilon_{i_1,\dots,i_N} U^{i_1}{}_{j_1} \dots U^{i_N}{}_{j_N} = \varepsilon_{j_1 \dots j_N}$

→ $\varepsilon_{i_1,\dots,i_N}$ is also an invariant tensor under SU(N)

→ For SU(3), $\varepsilon_{ijk} U^i{}_{i'} U^j{}_{j'} U^k{}_{k'} = \varepsilon_{i'j'k'}$, so that $\psi^i \chi^j \varepsilon_{ijk} = \psi^{[i}\chi^{j]} \varepsilon_{ijk}$

↳ $\phi^{ij} \varepsilon_{ijk} = \phi_A^{ij} \varepsilon_{ijk}$ is transforming in the conjugate rep.

We've shown that $(3 \otimes 3)_A = \bar{3}$

⊙ Mixed rank tensor:

Let us build an object $\psi^i \chi_j = \phi^i{}_j$. One can always separate the traceless part:

$$\psi^i \in N,\ \chi_j \in \bar{N};\quad \psi^i \chi_j = \phi^i{}_j = \left(\phi^i{}_j - \frac{\phi^k{}_k}{N}\delta^i{}_j\right) + \underbrace{\left(\frac{\phi^k{}_k}{N}\delta^i{}_j\right)}_{\text{in the trivial rep.}}$$

↳ One can write:

$$N \otimes \bar{N} = G \oplus 1 = (N^2-1) \oplus 1$$

In SU(3), $3 \otimes \bar{3} = 8 \oplus 1$

⊙ SU(5):

→ In SU(5), one has $\begin{cases} \psi^i \in 5 \\ \chi_i \in \bar{5} \end{cases}$, and $\begin{cases} \delta^i{}_j \\ \varepsilon_{ijk\ell m} \end{cases}$ invariants

↳ $5 \otimes \bar{5} = 24 \oplus 1$

→ Since $N(N-1)/2 \,|_{N=5} = 10$, $(5 \otimes 5)_A = 10$, and $5 \otimes 5 = 10_A \oplus 15_S$

5.3 Grand Unified Theories

→ A gauge theory is build on / a choice of gauge field
| a choice of matter representation

→ In the SM, the gauge group is $SU(3)_c \times SU(2)_L \times U(1)_Y$
The representation of matter fields are, among others,
→ the left-handed quark doublet $Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$ with quantum # $(\underset{SU(3)}{3}, \underset{SU(2)}{2}, \underset{Y/2}{1/6})_L$
→ the right-handed quarks u_R of Q# $(3, 1, 2/3)_R$
→ left handed lepton doublet $L_L = \begin{pmatrix} e_L \\ \nu_L \end{pmatrix}$ with Q# $(1, 2, -1/2)_L$ and L_L^c has $(1, 2, +1/2)$
→ right handed lepton singlet e_R with Q# $(1, 1, -1)_R$
↳ One family is $(3 \text{ colors} \times 2) + (3 \text{ colors} \times 1 \times 2) + (1 \text{ color} \times 2) + (1) = 15$ fields, and there are 3 families.

→ In GUTs, it is convenient to work with left-handed fields only. We then express (u_R, d_R, e_R) as:
$u_R \sim (3, 1, 2/3)_R \mapsto \bar{u}_L^c \sim (\bar{3}, 1, -2/3)_L$
$d_R \sim (3, 1, -1/3)_R \mapsto \bar{d}_L^c \sim (\bar{3}, 1, 1/3)_L$
$e_R \sim (1, 1, -1)_R \mapsto \bar{e}_L^c \sim (1, 1, 1)_L$

⊙ Including the SM in SU(5):

→ We consider $G = SU(5)$
→ $SU(5)$ and $SU(3) \times SU(2) \times U(1)$ have the same rank (number of independent diagonal (commuting) generators in its Cartan subalgebra). Indeed, $\text{Rank}[SU(3)] = 2$ $\text{Rank}[SU(2)] = 1$ and $\text{Rank}[U(1)] = 1$, so that $\text{Rank}[SM] = 4$.
The rank of $SU(N)$ is $\text{Rank}[SU(N)] = N-1$

→ $SU(5)$ has $5^2 - 1 = 24$ generators. In the fundamental rep, there are 5×5 traceless hermitian matrices. Let us organize the structure of the generators to reflect the subgroup structure of the SM.

→ SU(3) subgroup: 8 generators in a 3×3 matrix: $\begin{pmatrix} \boxed{3\times3} & 0 \\ 0 & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \end{pmatrix}$

→ SU(2) subgroup: 3 generators in a 2×2 matrix $\begin{pmatrix} \begin{smallmatrix} 0&0&0 \\ 0&0&0 \\ 0&0&0 \end{smallmatrix} & 0 \\ 0 & \boxed{2\times2} \end{pmatrix}$

↳ these 2 subgroups commute with each other.

→ A particular combination of diag. generators commutes with the 2 previous subgroups.

DEF The hypercharge is constructed using $\lambda^3 \sim \frac{1}{2}\begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}$ $\lambda^8 \sim \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}$ and $\sigma^3 \sim \frac{1}{2}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$, we chose normalization $\mathrm{Tr}\{(Y/2)^2\} = 1/2$:

$$\frac{Y}{2} \equiv \mathrm{diag}\left\{\frac{-1}{3}; \frac{-1}{3}; \frac{-1}{3}; \frac{1}{2}; \frac{1}{2}\right\}$$

→ Our 5-D object is then splited into $a = \{I, i\}$, $I = 1,2,3$ and $i = 1,2$:

$\psi = (\psi^a) = \begin{pmatrix} \psi^I \\ \psi^i \end{pmatrix}$. Since $\begin{cases} \psi^I \sim 1 \text{ under } SU(2) \text{ and} \\ \psi^i \sim 1 \text{ under } SU(3), \end{cases}$

we have $\psi^a \in 5 \sim (3,1,-1/3)_L \oplus (1,2,1/2)_L$

$$\bar{5} \sim \underbrace{(\bar{3},1,1/3)}_{d_L^c} \oplus \underbrace{(1,2,-1/2)}_{L_L}$$

↳ $(\psi^a) = \begin{pmatrix} d_R^r \\ d_R^g \\ d_R^b \\ \nu^c \\ e^c \end{pmatrix}_R$ and the conjugated field reads $(\psi_a) = \begin{pmatrix} d^r \\ d^g \\ d^b \\ \nu \\ e \end{pmatrix}_L$

→ Let us build an antisymmetric tensor $\phi_A^{ab} \equiv \psi^a \otimes_A \psi^b$

$$5 \otimes_A 5 \sim ((3,1,-1/3) \oplus (1,2,1/2)) \otimes_A ((3,1,-1/3) \oplus (1,2,1/2))$$

Now, $(3,1,-1/3) \otimes_A (3,1,-1/3) = (\bar{3}_A, 1, -2/3) \sim u^c$

↳ antiquark up

Also, $(1,2,1/2) \otimes_A (1,2,1/2) = (1, 1_A, 1) \sim e^c$

↳ antielectron

Finally, $(3,1,-1/3) \otimes_A (1,2,1/2) = (3,2,1/6) \sim Q_L$

↳ Since $5 \otimes 5 = \bar{5}_S \oplus 10_A$, one has

$$10 \sim (3,2,1/6) \oplus (\bar{3},1,-2/3) \oplus (1,1,1)$$

→ We've fitted all the SM reps (1 family) into $\bar{5} \oplus 10$. In components,

$$\psi_a \equiv \bar{5} = \begin{pmatrix} \psi_I \\ \psi_i \end{pmatrix} = \begin{pmatrix} d_R \\ d_G \\ d_B \\ \nu \\ e \end{pmatrix}_L \text{ and } \phi^{ab} \equiv 10 = \begin{pmatrix} \psi^{IJ} & \psi^{Ij} \\ \psi^{iJ} & \psi^{ij} \end{pmatrix} = \left(\begin{array}{ccc|cc} 0 & u^c & -u^c & u & d \\ -u^c & 0 & u^c & u & d \\ u^c & -u^c & 0 & u & d \\ \hline -u & -u & -u & 0 & e^c \\ -d & -d & -d & -e^c & 0 \end{array}\right)$$